

ON SOME EQUATIONS PERTAINING TO THE PROPAGATION OF HEAT IN AN INFINITE MEDIUM.

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(Plates XXIII-XXVIII.)

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We may attack a problem in the theory of the conduction of heat in two ways ; we may make use of a Fourier's series or integral, or, since the general differential equation is a partial linear one, we may build up the required solution out of known solutions for simpler cases. The former way is usually much the more expeditious if the proper "trick" can be hit upon, but the method is a purely artificial one, throwing no light on the process involved. The student or reader sees at once that this method produces the required result and that a limited number of very similar problems might be treated in the same way, but he is apt to feel instinctively at first that the mathematical tool he has employed is one of which he has only a superficial knowledge and that will fail him when he gets out of a certain set of problems ; he wonders what a Fourier's integral means and why it has a special value in such problems. The trouble here, as in many other departments of physics, is that the physical interpretation of mathematical operations is usually avoided. There can be but one good reason for this, since all must admit the desirability of such interpretations, that it is at times exceedingly difficult, if not impossible, to give the inherent physical meaning of a mathematical operation. Much more, however, might be done than is done, and there is perhaps no branch of mathematical physics more suited to the purpose of introducing to those just beginning such studies the meanings and the limitations of mathematical operations than heat conduction. The second method of treating heat conduction problems, by building up solutions from known solutions for other cases, is full of suggestiveness, and brings into view the meaning of many of the mathematical processes employed in any treatment of the conduction of heat, and the relationships of the equations involved. An attempt is made in the following pages to point out the necessity for effort along the lines indicated above, and among other things to give careful drawings of some of the more important curves of temperature and current.

In any heat conduction problem we have ordinarily three sets of equations, the general differential equation, the initial conditions, and the surface conditions. For the general purposes of this paper by taking the medium infinite we can get rid of the surface conditions without limiting the generality of the methods. Suppose we wish to study the case of a body of any shape or size maintained at any temperature in an infinite homogeneous medium of the same material as the body itself but initially at a uniform low temperature (which for convenience we take as the zero of temperature), or of the same body at a given initial temperature put into the medium and left to cool, we could find their solutions by an ordinary summation if we knew those for the corresponding problems in the case of an infinitesimally small particle. We might begin by assuming as Kelvin does (*Math. and Phys. Papers*, Vol. ii, p. 44), the solution for the case of a quantity of heat, Q , suddenly generated at a point $r = 0$ at time $t = 0$; but it will be better to see if it can be derived.

We have here to deal with the case of a symmetrical distribution of temperature about a point. The form of the general differential equation for this case is

$$\frac{1}{k} \frac{\partial V}{\partial t} = \frac{2}{r} \frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial r^2}, \dots\dots\dots (1)$$

where $k = \frac{K}{CD}$, K being the specific conductivity, C the specific heat, and D the density of the medium. This equation can be put in the more symmetrical form

$$\frac{1}{k} \frac{\partial(Vr)}{\partial t} = \frac{\partial^2(Vr)}{\partial r^2} \dots\dots\dots (2)$$

This is of exactly the same form as that for the case of the "linear flow of heat" of Fourier, that is, of flow in one dimension only, namely,

$$\frac{1}{k} \frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2} \dots\dots\dots (3)$$

The distribution of Vr with reference to r for the case of symmetry about a point is the same as the distribution of V with reference to x for the case of symmetry about an infinite plane perpendicular to the axis of x . This fact will be of assistance in obtaining and translating results. The ordinary way of treating

any problem of spherical symmetry is to get the simplest kind of a solution of (1) or (2) and build up from that solution to the required one. There is of course an infinite number of solutions of these equations and a great many simple ones, but we can at once find one by trying $Vr = e^{ar + \beta t}$. This gives $\beta = ka^2$, and hence $Vr = e^{ar} e^{ka^2 t}$. Changing a to ia we get $Vr = e^{-ka^2 t} (\cos ar + i \sin ar)$, and so a solution is

$$Vr = e^{-ka^2 t} \cos ar, \dots\dots\dots (4)$$

where a is any constant. This equation represents a periodic distribution of Vr along a radius vector dying out with the time; for the case of the infinite plane this would be actually the curve of distribution of temperature along x . It is seen that the values of V in (4) possess maxima and minima; the temperatures are zero at distances given by $r = (2n + 1) \frac{\pi}{2a}$ at all times. There is a hot central sphere of radius $\frac{\pi}{2a}$, surrounded by alternate hot and cold shells of common thickness $\frac{\pi}{a}$, the maximum numerical temperature in each falling as we go away from the centre. Calling the thickness of the shells d , we have $a = \frac{\pi}{d}$; so that the constant a is inversely proportional to the thickness of the shells and determines it. The central point begins by being, and remains, infinitely hot; the hot and cold layers conduct heat to each other and gradually die down in temperature. At a great distance from the origin we should have practically the case of a medium made up of alternate hot and cold infinite plates of the same numerical temperature and the same thickness left to cool; and such a problem could be treated from a consideration of (4).

This case is far from the problem we started out to discuss. We can, however, get new solutions from the simple one above, and the common method is now to say that the following is a solution of (2),

$$Vr = \int_0^{\infty} e^{-ka^2 t} \cos ar \, da, \dots\dots\dots (5)$$

and then translate this equation as we have just translated (4); but

instead of doing so we ought rather to be able to say that this operation means such and such and foretell the distribution of temperature it will give. This illustrates what was meant above when saying that we ought if possible to give the physical interpretations of mathematical processes. What is the meaning of the operation involved in (5)? Perhaps some light can be had on it from the following consideration: We are to take a series of distributions of temperature like that given by (4) and described above, where the constant a (determining the thickness of the shells) has the successive values, $0, da, 2da, \dots, a$, and superpose them on the medium after first reducing every temperature by multiplying it by da . We are then to take da indefinitely smaller and smaller, and finally to make a indefinitely greater and greater. We have thus the difficulty of a double limit entering, and if we wish to seek the initial condition it becomes a triple limit. This is sufficient to prevent any rash prediction in this problem as to the exact nature of the solution to be obtained; and this case serves as an excellent example of the difficulties to be overcome in any such efforts at physical interpretation. Before the limit is reached the state of temperatures is given by

$$Vr = da \left[1 + e^{-kt(da)^2} \cos ra + e^{-4kt(da)^2} \cos 2ra + \text{etc.} \right].$$

The limiting value of this series, which is equation (5), is not very evident without considerable study, but on account of the dying-out factor in each term the series is convergent, and the more rapidly convergent the greater the value of t , and its value could be found for any given t and da . Another way of finding this value at any time and distance required is to take an axis along

which a 's are measured and draw the logarithmic curve e^{-kta^2} and the curve $\cos ra$, then form the curve whose ordinate at each point is the product of the ordinates of these two curves at the point, and the area between this new curve and the axis gives the numerical value of Vr . Since this area is formed of pieces alternately above and below the axis of a and of decreasing numerical value, we see that Vr is always of the same sign and that, for any finite value of r , it begins by increasing in value and finally falls off to zero, and by inference that it is zero at time $t = 0$; but that at the origin it has initially a value greater than zero. The

operation (5) therefore promises at least another simple solution and one much nearer the desired one. Noting that

$$\int_{-\infty}^{+\infty} e^{-ka^2t} \cos ar \, da = 2 \int_0^{\infty} e^{-ka^2t} \cos ar \, da, \text{ and that}$$

$$\int_{-\infty}^{+\infty} e^{-ka^2t} \sin ar \, da = 0, \text{ we get } \int_{-\infty}^{+\infty} e^{-(kta^2 - ira)} \, da =$$

$$e^{-\frac{r^2}{4kt}} \int_{-\infty}^{+\infty} e^{-\left(\sqrt{kt}a - \frac{ir}{2\sqrt{kt}}\right)^2} \, da = \sqrt{\frac{\pi}{kt}} e^{-\frac{r^2}{4kt}},$$

and (5) becomes

$$Vr = \frac{A}{\sqrt{kt}} e^{-\frac{r^2}{4kt}}, \dots\dots\dots (6)$$

where A is an arbitrary constant. This equation says that Vr is initially indeterminate (evidently infinite, from physical considerations) at the centre and zero elsewhere; as time goes on the value of Vr falls off indefinitely at the centre, rises to a maximum at all other points and then falls off indefinitely also. Now these are exactly the conditions we want for V itself for the case of an infinitely hot point cooling in an infinite medium initially of zero temperature. If we had been studying (3) we would have found the same equation as (6), with x for r and V for Vr , for an infinitely hot plane cooling in a medium initially zero. The form of the curves for Vr given by (6) is exhibited on Plates XXIII and XXIV; with values of r as abscissæ curves A^1 to A^4 are for values of the time $\frac{1}{16k}$, $\frac{1}{8k}$, $\frac{1}{4k}$ and $\frac{1}{2k}$ respectively; with values of $4kt$ as abscissæ curves B^1 to B^5 are for values of the distance 0 , $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$ and 1 respectively.

We have taken the form (2) of the differential equation in preference to (1) on account of its symmetry and because we are solving the case of the infinite plane at the same time; but it possesses another important advantage. Since either form of the equation is a linear partial one we can add any number of solutions for a new solution; the question arises, therefore, whether V being a solution $\frac{\partial V}{\partial r}$ and $\int Vdr$ are solutions, and what are their physical

meanings. Without thinking of the special form of the differential equation, we can find the meaning of $\frac{\partial V}{\partial r}$ as follows: Let a solution, V , be $f(r, t)$; then another, V_1 , is $\frac{1}{\Delta r} f(r, t)$, where Δr is a small constant; and another, V_2 , is $-\frac{1}{\Delta r} f(r, t)$. Superpose on the medium these two states of temperature, V_1 and V_2 , after first displacing V_2 bodily to the positive side of the origin by an amount Δr . When Δr is indefinitely decreased the limiting state of temperature is that represented by $\frac{\partial V}{\partial r}$, or $\frac{\partial f(r, t)}{\partial r}$. That is, $\frac{\partial V}{\partial r}$ represents a heating due to a kind of doublet. We must next find out whether such a state of temperature as that represented by $\frac{\partial V}{\partial r}$ is a solution of (1). We saw that $\frac{\partial V}{\partial r}$ was a limiting case, and hence it is not a solution in the limit (except by some unusual accident) unless it is so just before the limit is reached. While Δr is still finite, but as small as we please, the superposed heatings do not satisfy the same differential equation; for V_1 satisfies the equation $\frac{1}{k} \frac{\partial f(r, t)}{\partial t} = \frac{2}{r} \frac{\partial f(r, t)}{\partial r} + \frac{\partial^2 f(r, t)}{\partial r^2}$, while V_2 satisfies the equation $\frac{1}{k} \frac{\partial f(r - \Delta r, t)}{\partial t} = \frac{2}{r - \Delta r} \frac{\partial f(r - \Delta r, t)}{\partial r} + \frac{\partial^2 f(r - \Delta r, t)}{\partial r^2}$, and on account of the variable coefficient these are not the same equation. Hence $\frac{\partial V}{\partial r}$ is not a solution of (1), and is only a solution of an equation in V when that equation has constant coefficients, that is, coefficients not containing r . Equation (2) is of that kind, and hence knowing a solution of it, V_r , we can say that $\frac{\partial(V_r)}{\partial r}$ is also a solution. Call this new solution $V_1 r$, then V_1 is a solution of (1). Since $\frac{\partial(V_r)}{\partial r} = V + r \frac{\partial V}{\partial r}$, and since $\frac{1}{r} \frac{\partial(V_r)}{\partial r}$ is a solution of (1), we have $\frac{V}{r} + \frac{\partial V}{\partial r}$ a solution of (1); this is what we have just called V_1 . Now V satisfies (1), but we have just seen that $\frac{\partial V}{\partial r}$ does not, and it can easily be seen that $\frac{V}{r}$ does not in general; so we have the interesting fact that the solution V_1 is the sum of two functions of V (itself a solution) neither of which is a solution. We can at least give a physical interpretation to the method of finding

a solution of (1) represented by the mathematical operation $\frac{1}{r} \frac{\partial(Vr)}{\partial r}$, where Vr is a solution of (2) and V itself a solution of (1); we have but to add to the doublet of this V as defined above a heating at each point r , which is V divided by the value of r at the point.

The meaning of $\int V dr$, where V is a solution of the differential equation, is now plain. It simply means finding a new function of r and t , V^1 , whose doublet is the solution V . That is, $\frac{\partial V^1}{\partial r} = V$, and $V^1 = \int V dr$. This is subject to the same limitations as before, that the differential equation for V must have its coefficients independent of r , in order that V^1 may be a solution of the equation.

Similarly for equation (2); we have a solution, Vr , to find the meaning of the new solution, V^1r , which we get on performing the integration $\int Vr dr$. Since $\frac{\partial(V^1r)}{\partial r} = Vr$, or $\frac{1}{r} \frac{\partial(V^1r)}{\partial r} = V$, we are but finding the distribution of temperature, V^1 , whose doublet added to the heating $\frac{V^1}{r}$ gives the distribution of temperature, V , which we started with.

We thus see that (2) has the great advantage over (1) that when we find a solution of the former we can differentiate and integrate it with regard to r for new solutions, but we cannot do so with the latter.

The meaning of $\frac{\partial V}{\partial t}$ and of $\int V dt$ as solutions of (1) are of the same general nature as the similar expressions with r , and are quite evident; we now superpose one heating, $\frac{1}{\Delta t} f(r, t)$ on another, $-\frac{1}{\Delta t} f(r, t)$, after a small interval of time Δt , which we make smaller and smaller indefinitely. We might call this a *time* doublet and the former a *space* doublet. Both $\frac{\partial V}{\partial t}$ and $\int V dt$ are solutions of (1) because the coefficients do not contain t . The same remarks apply to (2) as regards Vr , with the explanations of the former paragraph added. Here equation (2) possesses no advantage over (1).

The meaning of a Fourier's integral may now be given. A solution of (3) for the flow of heat in one dimension is evidently

$V = e^{-ka^2t} \cos \beta(a-x)$, where a and β are arbitrary constants, for it is made up of $V = Ae^{-ka^2t} \cos ax$ and $V = Be^{-ka^2t} \sin ax$, both of which are solutions of (3) as shown above. This equation denotes a distribution of temperatures which has maxima and minima values, the latter being at certain fixed points given by the equation $x = a - (2n + 1) \frac{\pi}{2\beta}$. In general it is very similar to the distribution represented by (4) already studied. $V_1 = V \varphi(a)$ is also a solution, where the temperatures are as before except that they are increased by multiplying every one by $\varphi(a)$, an arbitrary constant function of a . Another solution is got, as described before, by superposing all the heatings formed on reducing the temperatures in V_1 by multiplying each by the very small quantity da , and giving a all values from $-\infty$ to $+\infty$, and then taking the limiting case where da tends to zero. Call this new solution V_2 ;

then $V_2 = \int_{-\infty}^{+\infty} e^{-ka^2t} \cos \beta(a-x) \varphi(a) da$. Repeat this last operation

with regard to β ; that is, take the distribution of temperatures represented by V_2 and reduce the numerical value of each by multiplying by $d\beta$, then superpose all such heatings formed by giving β every value from 0 to ∞ , and finally take the limiting case where $d\beta$ tends to zero. Call this new solution V_3 ; then

$$V_3 = \int_0^{\infty} d\beta \int_{-\infty}^{+\infty} e^{-ka^2t} \cos \beta(a-x) \varphi(a) da.$$

Still another solution is got by reducing every temperature in V_3 in the ratio of π to 1.

Call this solution V_4 ; then $V_4 = \frac{1}{\pi} \int_0^{\infty} d\beta \int_{-\infty}^{+\infty} e^{-ka^2t} \cos \beta(a-x) \varphi(a) da$;

it has the special importance and peculiarity, as was first shown by Fourier, that at time zero the distribution of temperature it represents is the same function of x , $\varphi(x)$, that we took originally of a . Similarly every Fourier integral may be interpreted.

Returning now to equation (6) and the curves drawn for it, we can find new solutions by addition; at each point r let us add the temperature for that point and all other points farther from the centre, even to infinity, but first reduced in absolute value by multiplying each by the small quantity dr , which we make ultimately tend to zero. We have but to add on Plate XXIV for any

abscissa (time) the ordinates of all possible curves such as B^1 , B^2 , etc., below any given one, after reducing them as described. For $t = 0$ and $r = 0$ we would get $(\infty + 0 + 0 + \text{etc.}) dr$, which as dr diminishes indefinitely gives us some finite value; for other values of r we would get $(0 + 0 + 0 + \text{etc.}) dr$, which is zero. From the way the curves tend to become parallel it is suggested, and by trial we find, that for $r = 0$ and any finite value of the time not zero the sum of all the ordinates would be constant. We have then the promise of another simple solution, and can foretell its type somewhat, of the form

$$Vr = \int_r^\infty \frac{A}{\sqrt{kt}} e^{-\frac{r^2}{4kt}} dr = \frac{2}{\sqrt{\pi}} B \int_{\frac{r}{2\sqrt{kt}}}^\infty e^{-\beta^2} d\beta, \dots\dots(7)$$

where B is an arbitrary constant. On studying this equation we find that Vr at the origin has initially the value B , and maintains that value; at all other points it is initially zero and rises asymptotically with time toward the value B . V itself would be always infinite at the origin and initially zero elsewhere. For the case of linear flow equation (7) represents an infinite plane kept at temperature B in an infinite medium initially zero in temperature.

We can get the solution for an infinitely hot point put into an infinite medium initially zero and left to cool as follows: At time zero apply to the medium the state of temperatures represented by (7) with every temperature increased by multiplying it by the large quantity $\frac{1}{\Delta}$; after time Δt apply also the state of temperatures represented by (7) with sign changed and increased numerically as before; finally make Δt tend to zero. We have seen above that this is equivalent to performing the mathematical operation of differentiation of (7) with regard to t , that is, taking the time doublet of Vr . The reason that this solution is the one required is that the superposition of the two heatings gives Vr a large value at the origin at first and everywhere else a zero value, and then instantaneously makes Vr zero at the origin; that is, at the origin V is initially infinite in temperature and then falls off indefinitely, while all other points begin at zero and rise gradually. These were the conditions we wanted. Hence we have the solution

$$Vr = \frac{\partial}{\partial t} \left[\frac{2}{\sqrt{\pi}} B \int_0^{\frac{r}{2\sqrt{k}t}} e^{-\beta^2} d\beta \right] = \frac{Er}{(kt)^{\frac{3}{2}}} e^{-\frac{r^2}{4kt}}, \dots (8)$$

and

$$V = \frac{E}{(kt)^{\frac{3}{2}}} e^{-\frac{r^2}{4kt}}, \dots (9)$$

where E is an arbitrary constant.

Further light can be thrown on this problem by arriving at equation (8) by other methods. Remembering that equation (6) gave Vr initially infinite at the centre and zero elsewhere, and falling in value at the centre and gradually rising to a maximum elsewhere, we see that by taking the space doublet of this Vr we get Vr at the origin first infinite and then zero; that is, V at the origin is at first infinite and then gradually falls off, and is initially zero elsewhere and rises with time. These are the conditions required. Hence the solution is

$$Vr = \frac{\partial}{\partial r} \left[\frac{A}{\sqrt{k}t} e^{-\frac{r^2}{4kt}} \right] = \frac{Er}{(kt)^{\frac{3}{2}}} e^{-\frac{r^2}{4kt}} \dots (10)$$

Or we can look at it in this way: We saw that Vr in (6) had exactly the set of values we want V to have in the problem proposed, and the form of the right-hand member of (6), containing

as it does r in the factor $e^{-\frac{r^2}{4kt}}$ only, suggests at once that we can get the desired value of V by a simple differentiation with regard to r . This is what we have just done with a good physical reason for the operation.

Or another method. We saw that equation (6) for the case of flow in one direction only was that of an infinitely hot plane cooling in an infinite medium initially zero in temperature, and to get the solution for the similar problem in three dimensions we have but to multiply that solution by two similar ones with y and z substituted for x . This gives

$$V = E \frac{1}{(kt)^{\frac{3}{2}}} e^{-\frac{x^2}{4kt}} e^{-\frac{y^2}{4kt}} e^{-\frac{z^2}{4kt}} = \frac{E}{(kt)^{\frac{3}{2}}} e^{-\frac{r^2}{4kt}} \dots (11)$$

The rate of cooling is given by the equation

$$\frac{dV}{dt} = \frac{E}{2k^{\frac{3}{2}}t^{\frac{5}{2}}} \left(r^2 - 3 \right) e^{-\frac{r^2}{4kt}}.$$

Each point of the mass not the centre begins by being zero in temperature, then rises to a maximum after a time $t = \frac{r^2}{6k}$, and after this falls off indefinitely toward zero. The forms of the curves given by (9) are exhibited on Plates XXV and XXVI. With values of r as abscissæ curves I¹ to IV¹ are for values of the time $\frac{1}{16k}$, $\frac{1}{8k}$, $\frac{1}{4k}$, and $\frac{1}{2k}$ respectively; with values of $4kt$ as abscissæ curves 1¹ to 5¹ are for values of the distance 0, $\frac{1}{2}$, 1, $\frac{3}{2}$ and 2 respectively.

The meaning of the constant E is determined by finding the amount of heat supplied initially to the hot point. We have

$$Q = \iiint CDV dx dy dz = \frac{4\pi CDE}{(kt)^{\frac{3}{2}}} \int_0^\infty e^{-\frac{r^2}{4kt}} r^2 dr = 8CDE\pi^{\frac{3}{2}} \dots (12)$$

If we take as our unit of heat that required to raise the mass in a unit of volume of the substance 1°, the total quantity of heat, σ , in these units is

$$\sigma = 8E\pi^{\frac{3}{2}} \dots \dots \dots (13)$$

We could also get the total heat by taking the integral

$$\int_0^\infty -K \frac{\partial V}{\partial r} 4\pi r^2 dt. \text{ We get from (12) and (13) our equation (11)}$$

in the form

$$V = \frac{Q}{8CD(\pi kt)^{\frac{3}{2}}} e^{-\frac{r^2}{4kt}} = \frac{\sigma}{8(\pi kt)^{\frac{3}{2}}} e^{-\frac{r^2}{4kt}} \dots \dots \dots (14)$$

(See Kelvin's Papers, Vol. II, p. 44.)

We cannot build up by summation the solution for the case of a body of finite dimensions from the above solution for a mathematical point. We wish to pass to a case which has a physical significance, namely, a finitely hot particle left to cool in an infinite

medium of temperature initially zero. We can get a close approximation to this problem by putting the same quantity of heat, σ , into a particle of volume Δv which we put into the mathematical point, and assuming that the state of temperature produced in the surrounding medium is the same as that due to the infinitely hot point and is given accordingly by (14). This equation will represent the real state the better the longer the time which has elapsed, in accordance with the fact emphasized by Fourier that the initial heating is of less and less importance as the time is prolonged. The closeness of the approximation for any given time and distance will be brought out later.

Let the quantity of heat supplied raise the volume Δv to the temperature V_0 ; then $Q = CDV_0\Delta v$, or $\sigma = V_0\Delta v$; and (14) becomes

$$V = \frac{V_0\Delta v}{8(\pi kt)^{\frac{3}{2}}} e^{-\frac{r^2}{4kt}} \dots\dots\dots (15)$$

If the volume Δv is in the form of a sphere of radius R , (15) becomes

$$V = \frac{V_0 R^3}{6\sqrt{\pi}} \frac{1}{(kt)^{\frac{3}{2}}} e^{-\frac{r^2}{4kt}}, \dots\dots\dots (16)$$

and it is really for this form of the equation, with R taken as the unit of length, that the curves referred to on Plates XXV and XXVI were drawn. They are, as said, approximations only to the true curves. The latter may be found by the aid of a Fourier's integral. We know that the solution of (2) subject to the condition $V=f(r)$ when $t=0$ is

$$Vr = \frac{1}{\sqrt{\pi}} \left[\int_{-\frac{r}{2\sqrt{kt}}}^{\infty} (r + 2\sqrt{kt}\gamma) f(r + 2\sqrt{kt}\gamma) e^{-\gamma^2} d\gamma - \int_{\frac{r}{2\sqrt{kt}}}^{\infty} (-r + 2\sqrt{kt}\gamma) f(-r + 2\sqrt{kt}\gamma) e^{-\gamma^2} d\gamma \right] \dots (17)$$

Giving $f(r)$ the value V_0 from $r=0$ to $r=R$, and the value 0 from $r=R$ to $r=\infty$, (17) takes the form

$$V = \frac{V_0}{\sqrt{\pi}} \left[\int_{\frac{r-R}{2\sqrt{kt}}}^{\frac{r+R}{2\sqrt{kt}}} e^{-\gamma^2} d\gamma - \frac{\sqrt{kt}}{r} \left\{ e^{-\frac{(r-R)^2}{4kt}} - e^{-\frac{(r+R)^2}{4kt}} \right\} \right] \quad (18)$$

This then is the exact equation for a sphere of any size of initial temperature V_0 put into an infinite medium of the same material as the sphere of initial temperature zero and left to cool there. The forms of the curves given by this equation are exhibited on Plates XXV and XXVI, along with those of the approximate equation (16). Curves I to IV correspond to I¹ to IV¹, and curves 1 to 5 correspond to 1¹ to 5¹.

We can get an approximate form from equation (18) by expanding it in terms of R ; we find

$$V = \frac{V_0 R^3}{6\sqrt{\pi}} \frac{1}{(kt)^{\frac{3}{2}}} e^{-\frac{r^2}{4kt}} \left[1 + \frac{\frac{r^2}{kt} - 6}{40} \frac{R^2}{kt} \right] \dots\dots (19)$$

The first term of this is the same as equation (16), found otherwise. Equation (19) gives us a second approximation, and the second term within the bracket will enable us to determine the closeness of (16) as an approximation. In a similar problem, Fourier (Free-man's translation, p. 380) gives a limit to the time when the approximation may be used, but he does not give any means of telling how great the error is in general, and it was for the purpose of bringing this out distinctly that equation (19) and the curves on Plates XXV and XXVI were produced. From Plate XXV we see that the approximate curves are at first steeper and afterward flatter than the exact curves; they make the temperatures too high for points nearer the origin than a certain distance, and too low for points farther away. Indeed curves I and I¹ are very little alike for any value of r . As the value of the time for which the curve is drawn is taken greater and greater the curves approach each other more and more nearly, even for points less distant than unity (which are inside the little sphere), for which we might have expected little agreement. This makes evident the fact to which Fourier calls attention at the place just cited; one is very apt to assume that the curves would approach each other more and more as r is taken greater and greater, no matter what the value of t ; but just the reverse is true,

the curves approach each other more and more for greater and greater values of the time, no matter what the distance. This is seen more distinctly from an examination of Plate XXVI. There it will be seen also that the approximate curves are slower in reaching their maximum values, as well as that they have different maxima. For distances less than unity the approximate curves start at ∞ , while the exact curves start at $V = V_0$; for the distance unity the exact curve starts abruptly at $\frac{V_0}{2}$, while the approximate curve starts at 0 then gradually rises and has a maximum value less than $\frac{V_0}{3}$. For distances greater than unity both curves start at the origin.

From an inspection of the second term of (19) we can foretell the approximate accuracy of (16). Taking R as the unit of length, if $kt < 15$ the error in the value of $\frac{V}{V_0}$ will be everywhere greater than 1% except in the immediate neighborhood of $r = \sqrt{6kt}$, at which point the error is practically zero. For instance, for $kt = \frac{1}{2}$ (curves IV and IV') the approximate curve is 33% too high at $r = 0$, 22% at $r = 1$, correct at about 1.8, and 38% too low at 3. If $kt = 15$, the error is not more than 1% from $r = 0$ to $r = 13.4$. If $kt = 25$ the error is not more than 1% from $r = 0$ to $r = 20$. In general, for any value of kt the error is not more than 1% from $r = 0$ to $r = \sqrt{6kt + \frac{2}{3}(kt)^2}$, and from $r = 0$ to $r = \sqrt{6kt}$ the error decreases gradually from $\frac{15}{kt}\%$ to zero, and after that increases again. If we want results accurate to .01%, kt must be at least 1500, and in general for any value of kt greater than this the error is not more than .01% from $r = 0$ to $r = \sqrt{6kt + \frac{1}{250}(kt)^2}$, and from $r = 0$ to $r = \sqrt{6kt}$ the error decreases gradually from $\frac{15}{kt}\%$ to zero, and after that increases again.

From equation (15) we can build up by summation the equation for the case of a body of any shape or size initially at V_0 cooling in an infinite medium initially zero. In order to bring out a very interesting difference between summation and integration we shall apply equation (15) to the case of an infinite space, one-half of which is initially at V_0 and the other half at zero, the two parts being separated by an infinite plane surface. We shall first have to find the solution for a plane lamina. Take the central plane of the lamina as the plane of yz , and the origin where a perpendicular

from the point P , at which we want to know the temperature, meets this plane. Call the length of this perpendicular x . Break up the lamina into concentric rings of radius ρ about this origin, and let the distance of every point in one of such rings from the point P be r and the thickness of the lamina Δx ; then we have

$$V = \frac{V_0}{8(\pi kt)^{\frac{3}{2}}} \int_0^\infty e^{-\frac{x^2 + \rho^2}{4kt}} 2\pi\rho \cdot \Delta x \cdot d\rho = \frac{V_0 \Delta x}{2(\pi kt)^{\frac{1}{2}}} e^{-\frac{x^2}{4kt}} \dots\dots\dots (20)$$

From the symmetry of the problem this is evidently a case of linear flow, and the solution must satisfy equation (3). Knowing this solution (we can get it otherwise), the solution for three dimensions given in (15) can be deduced; we have but to multiply the value of $\frac{V}{V_0}$ for the case of one dimension by two similar expressions with y and z respectively substituted for x .

The corresponding electrical problem is that of an infinite cable with no lateral loss by leakage touched for an instant to a condenser of potential V_0 . If there is lateral leakage equation (20) is still the solution of the electrical problem; V is then not the potential, but the potential can be derived easily from it, as is well known.

If Q or σ , according to the unit of heat used, is the amount of heat required to raise the mass of a section of the plate of unit area by V_0 degrees, then $Q = CDV_0 \Delta x$, or $\sigma = V_0 \Delta x$, and equation (20) becomes

$$V = \frac{Q}{2CD(\pi kt)^{\frac{1}{2}}} e^{-\frac{x^2}{4kt}} = \frac{\sigma}{2(\pi kt)^{\frac{1}{2}}} e^{-\frac{x^2}{4kt}} \dots\dots\dots (21)$$

Of course this equation is of only the same grade of approximation as (15). It will be the more nearly exact the smaller Δx and, since the product of V_0 and Δx measures the heat in a section of unit area and is to remain constant, the greater V_0 . In the limit we should have the solution for an infinitely hot plane. The form of this solution we have already found; it is from (6) and the remarks following it

$$V = \frac{A}{V \sqrt{kt}} e^{-\frac{x^2}{4kt}} \dots\dots\dots (22)$$

Calling Q the total heat associated initially with a unit of area of

the plate, we find $Q = 2 \int_0^\infty CDV dx = 2ACD\sqrt{\pi}$; and this value of A reduces (22) to the form (21). Hence the general form of equation (21), which is approximate for a plate of actual thickness Δx , is exact for the infinitely hot plane. We shall revert to this important fact later.

If we want the exact equation for the plate of thickness Δx we can get it by the use of a Fourier integral. Making the obvious changes in (17) to suit it to the case of linear flow, and giving $f(x)$ the value V_0 from $x = -\frac{\Delta x}{2}$ to $x = \frac{\Delta x}{2}$ and the value 0 for all other values of x , we find

$$V = \frac{V_0}{\sqrt{\pi}} \int_{-\frac{x+\frac{\Delta x}{2}}{2\sqrt{kt}}}^{-\frac{x-\frac{\Delta x}{2}}{2\sqrt{kt}}} e^{-\gamma^2} d\gamma \dots\dots\dots (23)$$

Putting this in an approximate form, we have

$$V = \frac{V_0 \Delta x}{2(\pi kt)^{\frac{1}{2}}} e^{-\frac{x^2}{4kt}} \left[1 + \frac{x^2}{96 \frac{kt}{\Delta x}} \frac{(\Delta x)^2}{kt} \right], \dots\dots\dots (24)$$

the first term of which is equation (20). The forms of the curves for (20) are exhibited on Plates XXIII and XXIV. With values of x as abscissæ curves A^1 to A^4 are for values of the time $\frac{1}{16k}$, $\frac{1}{8k}$, $\frac{1}{4k}$ and $\frac{1}{2k}$ respectively; with values of $4kt$ as abscissæ curves B^1 to B^5 are for values of the distance 0, $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$ and 1 respectively. The second term of (24) enables us to tell approximately the degree of closeness of (20) to the exact equation (23). Taking Δx as the unit of length, if $kt < \frac{25}{12}$ the error will be everywhere greater than 1% except in the neighborhood of $x = \sqrt{2kt}$ where it is practically zero. If $kt = \frac{25}{12}$ the error is not more than 1% from $x = 0$ to $x = 2.9$, being 1% too high at $x = 0$, zero at $x = 2$, and 1% too low at $x = 2.9$. If $kt = 25$ the error is $\frac{1}{12}\%$ too high at $x = 0$, zero at 7, and 1% too low at 26. This is then a nearer approximation than the one discussed for the case of a hot particle, as was to

be expected. In general, for any value of kt the error is not more than 1% from $x=0$ to $x=\sqrt{2kt + \frac{24}{25}(kt)^2}$, and for any value of kt greater than $\frac{2500}{12}$ the error is not more than .01% from $x=0$ to $x=\sqrt{2kt + \frac{24}{2500}(kt)^2}$; from $x=0$ to $x=\sqrt{2kt}$ the error decreases gradually from $\frac{25}{12kt}\%$ to zero, and after that increases again.

The correspondingly approximate equation for the current or flow of heat in this case is

$$I = -K \frac{\partial V}{\partial x} = \frac{KV_0 \Delta x}{4\sqrt{\pi}} \frac{x}{(kt)^{\frac{3}{2}}} e^{-\frac{x^2}{4kt}} = \frac{K\sigma}{4\sqrt{\pi}} \frac{x}{(kt)^{\frac{3}{2}}} e^{-\frac{x^2}{4kt}} \dots (25)$$

The forms of these curves are given on Plates XXVII and XXVIII. With values of x as abscissæ curves C^1 and C_1^1 , C^2 and C_1^2 , and C_1^3 are for values of the time $\frac{1}{16k}$, $\frac{1}{8k}$ and $\frac{1}{4k}$ respectively; with values of $4kt$ as abscissæ curves D^1 and D_1^1 , D^2 and D_1^2 , and D_1^3 are for value of the distance $\frac{1}{4}$, $\frac{1}{2}$ and 1 respectively.

The exact equation for the flow, found from (23), is

$$I = \frac{KV_0}{2(\pi kt)^{\frac{1}{2}}} \left[e^{-\frac{(x - \frac{1}{2}\Delta x)^2}{4kt}} - e^{-\frac{(x + \frac{1}{2}\Delta x)^2}{4kt}} \right], \dots (26)$$

the curves for which have not been drawn.

By adding up the effects of an infinite number of such plates we can get the temperature due to one-half of space initially at a uniform temperature V_0 and the other half at zero temperature. Take the point P , at which the temperature is desired, in the cold half and at a distance x from the surface of separation, and take the origin in that surface at the foot of the perpendicular from P . Let one of the plates making up the other half of the medium be distant ξ from the origin. Then the x of equation (20) becomes $x + \xi$, and Δx becomes $\Delta \xi$; hence the temperature at P due to a series of such plates extending from $\xi=0$ to $\xi=\infty$, as found by integration, is

$$V = \frac{V_0}{2(\pi kt)^{\frac{1}{2}}} \int_0^\infty e^{-\frac{(x+\xi)^2}{4kt}} d\xi = \frac{V_0}{\sqrt{\pi}} \int_{\frac{x}{2\sqrt{kt}}}^\infty e^{-\beta^2} d\beta$$

$$= \frac{V_0}{2} \left[1 - \frac{2}{\sqrt{\pi}} \int_0^{\frac{x}{2\sqrt{kt}}} e^{-\beta^2} d\beta \right] \dots\dots\dots (27)$$

We could arrive at the solution for this case by using Fourier's integrals, as we did for equation (23), giving $f(x)$ the value V_0 from $x = -\infty$ to $x = 0$ and the value zero from $x = 0$ to $x = \infty$. We get at once equation (27) again.

This latter method gives the exact solution for the problem and yet it gives the same result as the former method, from which one might expect naturally enough an approximate solution, since we get it by integrating solutions that were approximate. This is the point to which attention was called in applying our results to this case; we have the integration of approximate solutions an exact solution. The first explanation offered of this unexpected result is apt to be that the approximation used is the more exact as the distance $x + \xi$ is the greater; but we have seen earlier that just the contrary is true and that at great distances (20) ceases to be properly called a solution unless the time is taken very great. The real explanation is simply that the operations of summation and integration are not always the same, and this is a case in point. Nothing is commoner in applying mathematics to physics than to use mathematical processes with laxity and to test the legitimacy of the application by the results. It is so uncommon to have a summation made improperly by integration that we lose sight of the mathematical fact that the operations are not equivalent. We take similarly the first two terms of a Taylor's series expansion as a sufficiently close approximation in almost any piece of analysis, without questioning whether the function under consideration can be so expanded and without reference to the value of the terms disregarded; we take differential coefficients without asking whether they can have a meaning, etc. The good excuse offered is that the chances are overwhelmingly in our favor, and that if we have made a mistake we shall quickly find it out from the results. Had we actually made a summation in the above problem we should have got an approximate result, but by integrating we get the limit toward which the summation tends as $d\xi$ tends towards zero, and it happens in this case that this is the exact solution. In finding an area we take a series of strips of area of $y dx$ and however infinitesimally small dx is, so long as it is something and not zero, the sum

of such strips is not the exact area required; $\int y dx$ is the limit toward which the sum tends as dx tends to zero, and we know from the familiar example of Fourier's series how the value can change actually in the limit. It happens in the present case that as $d\xi$ is made smaller and smaller, and V_0 correspondingly greater and greater in order to keep σ constant, in the limit $\frac{\sigma}{2(\pi kt)^{\frac{1}{2}}} e^{-\frac{x^2}{4kt}}$ is the exact solution for an infinite plane (see under (21) and (22)). So in making the integration above, that is, in finding the limit of the summation, we get necessarily an exact solution because in the limit each term of the solution is exact. Had we approached the limit in some other way than in keeping σ constant we might have got quite a different result.

The forms of the curves for (27) are shown on Plates XXVII and XXVIII. Curves E^1, E^2 and E^3 are drawn with values of x as abscissæ for values of the time $\frac{1}{16k}, \frac{1}{4k}$ and $\frac{1}{k}$ respectively; curves F^1, F^2 and F^3 are drawn with values of $4kt$ as abscissæ for values of the distance $\frac{1}{4}, \frac{1}{2}$, and 1 respectively.

Since the current or flow is got from the temperature by a differentiation with regard to x , and since equation (27) was got from (20) by an integration with regard to x , it is evident that the curves for the potential or temperature in (20) are the curves for current in the present problem.

$$I = -K \frac{\partial V}{\partial x} = \frac{KV_0}{2(\pi kt)^{\frac{1}{2}}} e^{-\frac{x^2}{4kt}} \dots\dots\dots (28)$$

These curves are given on Plates XXIII and XXIV for points to the right of the origin; the form for points to the left is obvious, since the curves are symmetrical about the yz plane.

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